$$E_k = \frac{a_k}{b_k - c_k E_{k-1}}, \tag{4.d.82}$$

and for $k = K$,

$$E_K = 0, \tag{4.d.83}$$

$$Fu_K = \frac{\{u^{n-1}\}_K + c_K Fu_{K-1}}{b_K - c_K E_{K-1}}. \tag{4.d.84}$$

In the above, the diagonal coefficient is given by

$$b_k = 1 + a_k + c_k. \tag{4.d.85}$$

Note that, since the model actually forecasts vorticity and divergence, the new diffused velocity fields u_k^{n+1} and v_k^{n+1} are not explicitly carried by the model, and the corresponding diffused vorticity and divergence are not calculated. Rather, the net effects of the diffusion $F_{u_V}\left(u^{n+1}\right)$ and $F_{v_V}\left(v^{n+1}\right)$ are saved for addition to the nonlinear terms n_U and n_V in the vorticity and divergence equations (after multiplication by $\cos\phi$),

$$F_{u_V} = \frac{u^{n+1} - u^{n-1}}{2\Delta t}, \tag{4.d.86}$$

$$F_{v_V} = \frac{v^{n+1} - v^{n-1}}{2\Delta t}. \tag{4.d.87}$$

The heating rate due to the diffusion is also calculated,

$$F_{T_V} = \frac{T^{n+1} - T^{n-1}}{2\Delta t}, \tag{4.d.88}$$

and added to the nonlinear term in the temperature tendency equation. The frictional heating associated with (4.d.86) and (4.d.87) is also added to the nonlinear term in the thermodynamic equation. The individual heating rates $F_{T_V}\left(T^{n+1}\right), -u^{n-1}F_{u_V}\left(u^{n+1}\right)/c_p^*$, and $-v^{n-1}F_{v_V}\left(v^{n+1}\right)/c_p^*$ are saved for diagnostic purposes. The new moisture q^{n+1} is retained as the basis for the next time–split step in the moisture equation. The moisture tendency due to vertical diffusion is also saved for diagnostic purposes,

$$F_{q_V}\left(q^{n+1}\right) = \frac{q^{n+1} - q^{n-1}}{2\Delta t}. \tag{4.d.89}$$

e. Gravity Wave Drag

Vertically propagating gravity waves can be excited in the atmosphere where stably stratified air flows over an irregular lower boundary and by internal heating and shear. These waves are capable of transporting significant quantities of horizontal momentum between their source regions and regions where they are absorbed or dissipated. Previous GCM results have shown that the large-scale momentum sinks resulting from breaking gravity waves play an important role in determining the structure of the large-scale flow, particularly for higher resolution truncations. The CCM3 incorporates a parameterization for a spectrum of vertically propagating internal gravity waves based on the work of Lindzen (1981), Holton (1982), Garcia and Solomon (1985) and MacFarlane (1987). Although the parameterization can treat a fairly general spectrum of monochromatic waves, in the standard tropospheric version of CCM3, the spectrum is reduced to a single stationary wave generated by flow over orography, following McFarlane (1987) and similar to the parameterization in CCM3. A more complete spectrum is used for middle atmosphere studies. As with all CCM3 physical parameterizations, non-resolvable-scale effects of vertically propagating gravity waves are determined in physical space on the transform grid.

Adiabiatic inviscid formulation

Following Lindzen (1981), the continous equations for the gravity wave parameterization are obtained from the two-dimensional hydrostatic momentum, continuity and thermodynamic equations in a vertical plane:

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) u = -\frac{\partial \Phi}{\partial x}\,, \tag{4.e.1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0\,, \tag{4.e.2}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) \frac{\partial \Phi}{\partial z} + N^2 w = 0\,. \tag{4.e.3}$$

Where N is the local Brunt-Väisällä frequency, and w is the vertical velocity in (log pressure) height coordinates. Eqs. (4.e.1)–(4.e.3) are linearized about a large scale background wind $\overline{u}$, with perturbations u', and combined to obtain:

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)^2 \frac{\partial^2 w'}{\partial z^2} + N^2 \frac{\partial^2 w'}{\partial x^2} = 0\,. \tag{4.e.4}$$

Solutions to (4.e.4) are assumed to be of the form:

$$w' = \hat{w} e^{ik(x-ct)} e^{z/2H} + \text{conjugate}\,, \tag{4.e.5}$$

where H is the scale height, k is the horizontal wavenumber and c is the phase speed of the wave. Subsituting (4.e.5) into (4.e.4), one obtains:

$$-k^2(\overline{u}-c)^2 \left(\frac{\partial}{\partial z} + \frac{1}{2H}\right)^2 \hat{w} - k^2 N^2 \hat{w} = 0\,. \tag{4.e.6}$$

Neglecting $\frac{1}{2H}$ compared to $\frac{\partial}{\partial z}$ in (4.e.6), one obtains the final form of the two dimensional wave equation:

$$\frac{d^2\hat{w}}{dz^2} + \lambda^2\hat{w} = 0\,, \tag{4.e.7}$$

with the coefficient defined as:

$$\lambda = \frac{N}{(\overline{u} - c)}\,. \tag{4.e.8}$$

The WKB solution of (4.e.8) is:

$$\hat{w} = A\lambda^{-1/2}\exp\left(i\int_0^z \lambda dz'\right)\,, \tag{4.e.9}$$

where A is determined from the wave amplitude at the source ($z = 0$), The Reynolds stress associated with the (4.e.9) is:

$$\tau(z) = \tau(z = 0) = \rho\overline{u'w'} = -\frac{2}{k}|A|^2\rho_0\mathrm{sgn}(\lambda)\,, \tag{4.e.10}$$

and is conserved while the momentum flux $\overline{u'w'}$ grows exponentially with height as the inverse of the density.

Saturation condition

The wave amplitude in (4.e.1) grows as $e^{z/2H}$ until the wave becomes unstable to convective overturning, Kelvin-Helmholtz instability, or other nonlinear processes. At that point, the wave amplitude is assumed to be controlled at the amplitude which would lead to the onset of the instability and the wave is "saturated". The saturation condition used in CCM3 is from McFarlane (1987), based on a maximum Froude number (F_c), or streamline slope.

$$|\rho\overline{u'w'}| \leq \tau^* = F_c^2 E\frac{k}{2}\rho\frac{|\overline{u} - c|^3}{N}\,, \tag{4.e.11}$$

where τ^* is the saturation stress, E is an "efficiency factor", and $F_c^2 E \sim 0.1$. Following Lindzen (1981), within a saturated region the momentum tendency can be determined analytically from the divergence of the saturation stress (4.e.11):

$$\begin{aligned}\frac{\partial u}{\partial t} &= -\frac{1}{\rho}\frac{\partial}{\partial z}\rho\overline{u'w'}\,,\\ &\simeq E\frac{k}{2}\frac{(\overline{u} - c)^3}{N}\frac{1}{\rho}\frac{\partial\rho}{\partial z}\,,\\ &\simeq E\frac{k}{2}\frac{(\overline{u} - c)^3}{N}\frac{g}{RT}\,,\end{aligned} \tag{4.e.12}$$

The analytic solution (4.e.12) is used below as an upper bound on the tendency computed by differentiating the stress profile. This condition is usually invoked when the wave saturates close to a critical level ($\overline{u} - c = 0$).

Radiative damping and dissipation

The effect of radative damping of the vertically propagating gravity waves can be included in the solutions (4.e.9) and (4.e.10) relatively easily through the Newtonian cooling approximation. The effect of mechanical mixing can also be included through an eddy viscosity, giving a damped vertical stress profile:

$$\tau(z) = \tau(0)\exp\left(-\frac{2}{H}\int_0^z \lambda_i dz'\right) , \tag{4.e.13}$$

where the imaginary part of the local vertical wavenumber, λ_i depends on the Newtonian cooling coefficient α and eddy viscosity D as:

$$\lambda_i = \frac{N}{2k(\overline{u}-c)^2}\left(\alpha + \frac{N^2}{(\overline{u}-c)^2}D\right) . \tag{4.e.14}$$

In CCM3, $\alpha = 10^{-6}$ s^{-1} and is not important in the solution. D is determined from the saturation condition, but is only of modest importance, particularly in the standard tropospheric model with only a single wave. Again following Lindzen (1981), one can assume that within a region of wave staturation, the wave damping balances the exponential growth with height. The required diffusivity is then:

$$D^* = \frac{(\overline{u}-c)^2}{N^2}\left[\frac{Ek}{HN}(\overline{u}-c)^2 - \alpha\right] . \tag{4.e.15}$$

We define D^* as the saturation diffusivity, analogous to the saturation stress τ^*, for use below.

Orographic source function

For orographically generated waves, the source is taken from McFarlane (1987):

$$\tau_g = |\rho\overline{u'w'}|_0 = E\frac{k}{2}h_0^2\rho_0 N_0 \overline{u}_0 , \tag{4.e.16}$$

where h_0 is the streamline displacement at the source level, and ρ_0, N_0, and $\overline{u}_0$ are also defined at the source level. For orographic waves, the subgrid-scale standard deviation of the orography σ is used to estimate the average mountain height, determining the typical streamline displacement. An upper bound is used on the displacement which corresponds to requiring that the wave not be supersaturated at the source level.

$$h_0 = \min(2\sigma, F_c\frac{\overline{u}_0}{N_0}) . \tag{4.e.17}$$

The source level quantities ρ_0, N_0, and $\overline{u}_0$ are defined by vertical averages over the source region, taken to be 2σ, the depth to which the average mountain penetrates into the domain:

$$\psi_0 = \int_0^{2\sigma} \psi\rho dz, \qquad \psi \in \{\rho, N, u, v\} . \tag{4.e.18}$$

The source level wind vector (u_0, v_0) determines the orientation of the coordinate system in (4.e.1)–(4.e.3) and the magnitude of the source wind $\overline{u}_0$.

Gravity wave spectrum

A small background gravity wave stress τ_b is assumed to be present at all points and the actual source stress is determined from the maximum of the orographic source (4.e.16) and the background value. When the background stress is used, the source level is assumed to be the bottom model level and the gravity wave stress at the source level is aligned with source wind, as usual. At all other levels, the local wind vector is projected onto the source wind vector, reducing the problem to two dimensions.

A gravity wave spectrum can be included by setting the number of waves to be greater than zero. In that case,

$$\tau_s(c_\ell) = \tau_b \exp\left[-\left(\frac{c_\ell}{30}\right)^2\right]$$
$$c_\ell \in [0, \pm 10, \pm 20, ...]\,. \tag{4.e.19}$$

Above the source region, the saturation condition is enforced separately for each wave:

$$\tau(c_\ell) \leq \tau_\ell^* = F_c^2 E \frac{k}{2} \rho \frac{|\overline{u} - c_\ell|^3}{N}\,, \tag{4.e.20}$$
$$\frac{\partial \overline{u}_\ell}{\partial t} = -\frac{1}{\rho}\frac{\partial \tau_\ell}{\partial z} \leq -\frac{1}{\rho}\frac{\partial \tau_\ell^*}{\partial z} = E\frac{k}{2}\frac{|\overline{u} - c_\ell|^3}{N}\frac{g}{RT}\,, \tag{4.e.21}$$
$$\frac{\partial \mathbf{V}}{\partial t} = \frac{\mathbf{V}_s}{|\mathbf{V}_s|}\sum_l \frac{\partial \overline{u}_\ell}{\partial t}\,. \tag{4.e.22}$$

Finally, the diffusivity used to determine the stress profile (4.e.13) outside of saturated regions is

$$D = \max\left(D_\ell^* \frac{\tau_\ell}{\tau_\ell^*}, \forall \ell\right)\,. \tag{4.e.23}$$

Although the standard CCM3 only uses a single gravity wave with $c = 0$, the code allows for the more general spectrum described above.

Numerical approximations

The gravity wave drag parameterization is applied immediately after the nonlinear vertical diffusion. The interface temperatures are first determined from:

$$T_{k+1/2}^{n-1} = \frac{T_k^{n-1} + T_{k+1}^{n-1}}{2},. \tag{4.e.24}$$

The interface Brunt-Väisällä frequency is

$$N_{k+1/2}^2 = \frac{g^2}{T_{k+1/2}^{n-1}} \left(\frac{1}{c_p} - \rho \frac{\partial T}{\partial p} \right)_{k+1/2}, \tag{4.e.25}$$

Where the interface density and temperature gradient are:

$$\rho_{k+1/2}^{n-1} = \frac{RT_{k+1/2}^{n-1}}{p_{k+1/2}}, \tag{4.e.26}$$

$$\left(\frac{\partial T^{n-1}}{\partial p} \right)_{k+1/2} = \frac{T_k^{n-1} - T_{k+1}^{n-1}}{p_k - p_{k+1}}. \tag{4.e.26}$$

The midpoint Brunt-Väisällä frequencies are determined as $N_k = (N_{k+1/2} + N_{k-1/2})/2$.

The level for the orographic source is an interface determined from an estimate of the vertical penetration of the subgrid mountains within the grid box. The subgrid scale standard deviation of the orography, σ_h, gives the variation of the mountains about the mean elevation, which defines the Earth's surface in the model. Therefore the source level is defined as the interface, $k_s - 1/2$, for which $z_{k_s+1/2} < 2\sigma_h < z_{k_s-1/2}$, where the interface heights are defined from the midpoint heights by $z_{k+1/2} = \sqrt{(z_k z_{k+1})}$.

The source level wind vector, density and Brunt-Väisällä frequency are determined by vertical integration over the region from the surface to interface $k_s + 1/2$:

$$\psi_0 = \sum_{k=k_s}^{K} \psi_k (p_{k+1/2} - p_{-1/2}), \qquad \psi \in \{\rho, N, u, v\}. \tag{4.e.27}$$

The source level background wind is $\overline{u}_0 = \sqrt{(u_0^2 + v_0^2)}$ and the unit vector for the source wind is

$$(x_0, y_0) = (u_0, v_0)/\overline{u}_0, \tag{4.e.28}$$

the projection of the midpoint winds onto the source wind is

$$\overline{u}_k = u_k^{n-1} x_0 + v_k^{n-1} y_0, \tag{4.e.29}$$

and the projected interface winds are

$$\overline{u}_{k+1/2} = (\overline{u}_k + \overline{u}_{k+1})/2. \tag{4.e.30}$$

Assuming that $\overline{u}_0 > 2$ m s^{-1} and $2\sigma^h > 10$ m, then the orographic source term, τ_g is given by (4.e.16) and (4.e.17), with F_c =0.5, $E = .125$, and $k = 2\pi/10^5$ m^{-1}. The gravity wave source term is then

$$\tau_0 = \max(\tau_g, \tau_b), \tag{4.e.31}$$

$$\tau_b = \tau_e + \tau_{(N,S)}, \tag{4.e.32}$$

where $\tau_N = 0.75 \sin |2\phi|$ and $\tau_S = 1.2 \sin |2\phi|$ are applied in the Northern and Southern Hemispheres, respectively, and $\tau_e = \exp\left[-(\phi/10)^2\right]$ Pa. The background values are not significant in tropospheric version of CCM3, but become important for middle atmosphere studies.

Although the code contains a provision for a linear stress profile within a "low level deposition region", this part of the code is not used in the standard model, since $\tau_{\rm ldv} = 0$.

The stress profiles are determined by scanning up from the bottom of the model to the top. The stress at the source level is determined by (4.e.31). The saturation stress, τ^* at each interface is determined by (4.e.11), where we note that $\tau^* = 0$ if a critical level is passed. A critical level is contained within a layer if $(\overline{u}_{k+1/2} - c)/(\overline{u}_{k-1/2} - c) < 0$. The diffusivity at each interface is determined by

$$D_{k-1/2} = D^*_{k-1/2} \frac{\tau_{k+1/2}}{\tau^*_{k-1/2}}, \tag{4.e.33}$$

where D^* is given by (4.e.15). The interface stress is then determined from the stress on the interface below by:

$$\tau_{k-1/2} = \min(\tau^*_{k-1/2}, \tau_{k+1/2} \exp\left[-2m_i \frac{R}{g} T_k (\ln p_{k+1/2} - \ln p_{k-1/2})\right], \tag{4.e.34}$$

Limits on the range of stresses within the spectrum are present in the code, but are only applied when the spectrum contains more than a single wave, so are not relevant to the standard model.

Once the complete stress profile has been obtained, the forcing of the background wind is determined by differentiating the profile during a downward scan:

$$\frac{\partial \overline{u}_k}{\partial t} = g \frac{\tau_{k+1/2} - \tau_{k-1/2}}{p_{k+1/2} - p_{k-1/2}} < \left(\frac{\partial \overline{u}_k}{\partial t}\right)^{\max}. \tag{4.e.35}$$

$$\left(\frac{\partial \overline{u}_k}{\partial t}\right)^{\max} = \min\left[\left(\frac{\partial \overline{u}_k}{\partial t}\right)^*, \frac{.5|c - \overline{u}_k|}{2\Delta t}, 500\mathrm{m/s}^{-1}/\mathrm{day}^{-1}\right]. \tag{4.e.36}$$

The first bound on the forcing, $(\partial \overline{u}_k / \partial t)^*$ is given by the analytic solution (4.e.12). This bound typically comes into play when a critical level is near and the background wind is rapidly varying in the vertical. The second bound comes from requiring that the forcing not be large enough to push the wind mode than half way towards a critical level within a time step and takes the place of an implicit solution. This bound is present for numerical stability, it comes into play when the time step is too large for the forcing. It is not feasible to change the time step, or to write an implicit solver, so an *a priori* bound is used instead. The third bound is used to constrain the forcing to lie within a physically plausible range (although the value used is extremely large) and is rarely invoked.

When any of the bounds in (4.e.35) are invoked, conservation of stress is violated. In this case, stress conservation is ensured by decreasing the stress on the lower interface to match the actual stress divergence in the layer:

$$\tau_{k+1/2} = \tau_{k-1/2} + \frac{\partial \overline{u}_k}{\partial t} \frac{p_{k+1/2} - p_{k-1/2}}{g}\,. \tag{4.e.37}$$

This has the effect of pushing some of the stress divergence into the layer below, a reasonable choice since the waves are propagating up from below.

Finally, the vector momentum forcing by the gravity waves is determined by projecting the background wind forcing with the unit vectors of the source wind:

$$\frac{\partial \mathbf{V}_k}{\partial t} = \frac{\partial \overline{u}_k}{\partial t}(x_0, y_0)\,. \tag{4.e.38}$$

The gravity wave source is also projected in the direction of the source wind for recording on the history files. The momentum tendencies from (4.e.38) are added to the nonlinear terms in the vorticity and divergence equations. In addition, the frictional heating implied by the momentum tendencies, $\frac{1}{c_p^*}\mathbf{V}_k^{n-1}\partial \mathbf{V}_k/\partial t$, is added to the thermodynamic equation.

f. Rayleigh Friction

The Rayleigh friction term in CCM3 is a carryover from CCM2. It was only used for the stratosphere and, for those applications, it has been superseded by the gravity wave spectrum parameterization described in the previous section. Since the parameterization is still in the CCM3 code, it is described here. However, the parameterization will probably be removed in revised versions of CCM3.

A Rayleigh friction term has been included in the zonal momentum equation to provide a crude parameterization of the effect of breaking gravity waves in the mesosphere and to prevent reflection of waves from the top boundary. This term is implemented in the form

$$\frac{\partial \mathbf{V}}{\partial t} = -K_R \mathbf{V} \tag{4.f.1}$$

The friction coefficient is determined from

$$K_R = \frac{1}{3}\left[1 + \tanh\left(\frac{z - 63 \times 10^3}{7.5 \times 10^3}\right)\right] \ \text{days}^{-1}, \tag{4.f.2}$$

where

$$z = -h \ln\left(A + B\right), \tag{4.f.3}$$

and $h = 7 \times 10^3$ m. This term, which is not intended to be a faithful representation of mesospheric dynamics is inactive in the standard model configuration.

4.2 Adjustment Physics

After the new temperatures and mixing ratios are computed from (3.a.5) and (3.a.6), a series of convective adjustment schemes is applied to create mutually adjusted